THE TWENTY-FIRST ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-first annual meeting of the Society was held in New York City on Friday and Saturday, January 1–2, 1915. The attendance at the four sessions included the following ninety-five members:

Mr. J. W. Alexander, II, Dr. E. S. Allen, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Mr. A. A. Bennett, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor B. H. Camp, Professor C. W. Cobb, Professor A. B. Coble, Dr. Emily Coddington, Professor A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor T. M. Focke, Professor A. B. Frizell, Professor O. E. Glenn, Professor J. W. Glover, Dr. G. M. Green, Professor T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor E. R. Hedrick, Dr. A. A. Himowich, Professor T. F. Holgate, Professor L. A. Howland, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor O. C. Lester, Professor Florence P. Lewis, Mr. P. H. Linehan, Professor W. R. Longley, Professor A. C. Lunn, Professor James Maclay, Professor Mansfield Merriman, Dr. E. J. Miles, Professor H. H. Mitchell, Dr. R. L. Moore, Dr. F. M. Morgan, Professor Frank Morley, Professor F. R. Moulton, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor E. D. Roe, Jr., Dr. Joseph Rosenbaum, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. L. L. Silverman, Professor H. E. Slaught, Professor P. F. Smith, Professor Virgil Snyder, Professor K. D. Swartzel, Professor Elijah Swift, Professor Evan Thomas,

Professor H. W. Tyler, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. J. N. Vedder, Mr. H. E. Webb, Professor C. J. West, Professor H. S. White, Professor E. E. Whitford, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair at the opening session, being relieved by Vice-President L. P. Eisenhart and after the annual election by the President-elect, Professor E. W. Brown, and Vice-President Oswald Veblen. The council announced the election of the following persons to membership in the Society: Dr. Florence E. Allen, University of Wisconsin; Dr. Nathan Altshiller, University of Washington; Dr. D. F. Barrow, University of Texas; Dr. R. B. Robbins, Sheffield Scientific School; Mr. C. H. Yeaton, University of Chicago. Fifteen applications for membership in the Society were received.

The total membership of the Society is now 709, including 69 life members. The total attendance of members at all meetings of the past year was 428; the number of papers read was 192. The number of members attending at least one meeting during the year was 255. At the annual election 210 votes were cast. The Treasurer's report shows a balance of $9,461.75, including the life membership fund of $5,198.85. Sales of the Society's publications during the year amounted to $1,843.67. The Library now contains about 5,100 bound volumes.

The annual meeting this year was especially marked as the occasion of the delivery of President Van Vleck's retiring address, the subject of which was "The rôle of the point set theory in geometry and dynamics." The address was given on Friday afternoon before an audience of nearly one hundred, a considerable number having come on from the West and from the American Association meeting at Philadelphia.

Another special feature of the meeting, on the social side, was the annual dinner and smoker held at the Yale Club on Friday evening with an attendance of seventy members.

At the annual election, which closed on Saturday morning, the following officers and other members of the Council were chosen:

President,	Professor E. W. Brown.
Vice-Presidents,	Professor F. R. Moulton, Professor Oswald Veblen.

Secretary, Professor F. N. COLE.
Treasurer, Professor J. H. TANNER.
Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1917,

Professor G. D. BIRKHOFF, Professor R. G. D. RICHARDSON,
Professor O. E. GLENN, Professor W. H. ROEVER.

Committees were appointed to consider the question of holding a colloquium in connection with the summer meeting at San Francisco, and the possible relations of the Society to the field now covered by the *American Mathematical Monthly*. A committee is also considering matters relating to the teaching of mathematics in the schools.

The following papers were read at this meeting:

(1) Professor L. P. EISENHART: "Transformations of surfaces Ω."

(2) Dr. L. L. SILVERMAN: "On the notion of summability for the limit of a function of a continuous variable."

(3) Professor A. B. COBLE: "A configuration in finite geometry."

(4) Professor A. B. COBLE: "The elliptic norm curve in S_4."

(5) Professor J. E. ROWE: "The symbolic and actual form of certain combinants of two binary n-ics."

(6) Professor ARTHUR RANUM: "On the differential geometry of the cyclic (circled) surfaces."

(7) Professor A. B. FRIZELL: "An enumeration of integral algebraic polynomials."

(8) Dr. DUNHAM JACKSON: "Expansion problems with irregular boundary conditions."

(9) Dr. G. M. GREEN: "Hypersurfaces and families of curves defined by solutions of a partial differential equation of the second order."

(10) Dr. CAROLINE E. SEELY: "Certain non-linear integral equations."

(11) President E. B. VAN VLECK, Presidential address: "The rôle of the point set theory in geometry and dynamics."

(12) Professor W. F. OSGOOD: "On the division of space of n dimensions by a simple closed manifold."

(13) Professor G. D. BIRKHOFF: "The functions of several variables defined by linear difference and differential equations."

(14) Professor G. D. BIRKHOFF: "Note on the reducibility of maps."

(15) Professors VIRGIL SNYDER and F. R. SHARPE: "Certain quartic surfaces belonging to infinite discontinuous cremonian groups."

(16) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem."

(17) Professor J. L. COOLIDGE: "Circular transformations and complex space."

(18) Professor G. A. MILLER: "Note on several theorems due to A. Capelli."

(19) Professor EDWARD KASNER: "The generalized concept of differential element."

(20) Professor F. N. COLE: "Note on solvable quintics."

(21) Professor G. C. EVANS: "Note on the variation of a function depending on all the values of another function."

(22) Professor E. V. HUNTINGTON: "A set of postulates for elementary dynamics" (preliminary communication).

(23) Professor F. R. MOULTON: "The solution of an infinite system of implicit functions, with an application to Hill's lunar theory."

(24) Professor C. N. HASKINS: "On the roots of the incomplete gamma function."

(25) Dr. W. C. GRAUSTEIN: "On the geodesics and geodesic circles on a developable surface."

(26) Dr. D. F. BARROW: "Oriented circles in space."

(27) Professor JAMES MACLAY: "A transformation of polynomials relative to the exponents."

(28) Mr. J. W. ALEXANDER, II: "A method for resolving the singularities of algebraic manifolds."

(29) Professor T. H. GRONWALL: "On the summation method of de la Vallée-Poussin."

(30) Professor T. H. GRONWALL: "An integral equation of the Volterra type."

(31) Professor T. H. GRONWALL: "On the distortion in conformal representation."

The papers of Professor Rowe, Dr. Seely, Mr. Vandiver, Professor Miller, Professor Cole, Dr. Graustein, Dr. Barrow, and the last two papers of Professor Gronwall were read by

title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Professor Eisenhart showed that by an adaptation of the theorem of Moutard for differential equations of the Laplace type with equal invariants it is possible to transform a surface S, referred to a conjugate system with equal point invariants, into a surface S_1 upon which the parametric curves are of the same sort. This transformation, which he called a transformation K, is such that for the congruence of lines joining corresponding points on S and S_1 the developables cut these surfaces in the parametric system and the focal points on any line are harmonic with respect to the points of S and S_1 on the line. In the present memoir he considers the particular case where this congruence is normal. Surfaces orthogonal to such a congruence have been considered by Demoulin, who called them surfaces Ω. A surface S which admits a transform S_1 such that the congruence is normal is characterized by the property that the conjugate system with equal invariants is $2, 0$, to use the notation of Guichard. When such a surface is given, the determination of the normal congruence is a question of quadratures. With the aid of the theorem of permutability of transformations K applied to these surfaces S, it is possible to establish transformations A of surfaces Ω into surfaces Ω such that a surface and a transform are the envelope of a two-parameter family of spheres with the lines of curvature corresponding on the two surfaces. These transformations also admit a theorem of permutability.

Isothermic surfaces are surfaces Ω, and the transformations A which give rise to new isothermic surfaces are the transformations D_m, discovered by Darboux and studied at length by Bianchi. Surfaces with isothermal spherical representation of their lines of curvature are surfaces Ω with one of the surfaces S at infinity, and consequently require special study. When the surfaces resulting from such a surface by transformations A are surfaces of the same sort, the transformations are the same as those established by the author some years ago from a different point of view.

2. A sequence of numbers may be regarded as a function $u(n)$ of a discrete variable n. Toeplitz has defined a related sequence

$$v(n) = \sum_{i=1}^{i=\infty} k(n, i)u(i),$$

where $k(n, i)$ is a function of two discrete variables, and has obtained conditions on $k(n, i)$ which are necessary and sufficient that the existence of the limit of $u(n)$ imply the existence of the limit of $v(n)$ and the equality of the two limits. Replacing the discrete variables by continuous variables and the sign of summation by an integral sign, Dr. Silverman considers the related functions $u(x)$ and

$$v(x) = \int_a^\infty k(x, y)u(y)dy$$

and obtains conditions on $k(x, y)$ which are necessary and sufficient that the existence of the limit of $u(x)$ imply the existence of the limit of $v(x)$ and the equality of the two limits. The theorem obtained applies in particular to improper integrals.

3. For $p = 1, 2, 3$ there are sets of even theta functions in p variables which are related to a set of $2p + 2$ points in a projective space S_p. The purpose of Professor Coble's first paper is to show that, in the finite geometry associated with the functions, there exist for any value of p configurations which are in tactical correspondence with such a set of points.

4. Among all the elliptic norm curves, that of order five in S_4 is peculiar in that the quadrics which contain the curve serve to define projectively a Cremona transformation. This transformation is studied in Professor Coble's second paper as an aid to the location of the point sets mentioned in the first paper.

5. The important rôle played by the combinants of two binary forms in the theory of rational plane curves is dependent upon the fact that if they are expressed in *combinant form* (i. e., in terms of the two-rowed determinants of the matrix of coefficients of the two binary forms) an easy translation scheme makes it possible to transform these combinants into the equations of covariant curves of rational plane curves. The purpose of Professor Rowe's paper is to derive a series of combinants of two binary n-ics which are not only

in combinant form, but which may be expressed as determinants of comparatively low order whose constituents are functions of the two-rowed determinants of the matrix of coefficients of the two binary forms; also, it is shown how these may be represented symbolically in such a way that the actual expression for a combinant of any two particular n-ics may be written out with facility. Possibly the simplest combinant of this type is the one whose vanishing is the necessary and sufficient condition that two binary n-ics have polar cubics with two common factors.

6. In the study of cyclic surfaces much attention has been given in the past to certain very special kinds, for instance to anallagmatic surfaces, including cyclides. In the theory of the general cyclic surface, very little progress has been made, and the few results obtained have been confined almost exclusively to metrical and conformal properties. The more fundamental descriptive properties, which are invariant under the 24-parameter group of sphere transformations, have hardly been touched. They form the subject of the present paper by Professor Ranum, which is based on two earlier papers on the projective differential geometry of n-space, one published in the *Annali* in 1912 and the other soon to appear in the *American Journal*. The results are obtained by employing the well-known correspondence between the projective line geometry of 4-space and the descriptive circle geometry of 3-space. It is shown that from this viewpoint cyclic surfaces naturally fall into eleven classes, of which four are annular and seven non-annular. Among the striking results is the fact that there exists on every non-annular surface (excepting only those which are anallagmatic) a certain fundamental pair of curves, whose importance for the surface is somewhat similar to that of the edge of regression for a developable surface.

7. By arranging the natural numbers as products of powers of primes, Professor Frizell makes an actual enumeration of the algebraic equations in which all coefficients are positive integers. Similarly the equations in which negative integers occur as coefficients are put into one-to-one correspondence with the set of positive fractions in their lowest terms. Then the two sets of equations may be assigned to the even and odd numbers respectively.

8. In most of the expansion problems which have been studied in detail, in connection with ordinary linear differential equations, the boundary conditions are of the sort which Birkhoff calls *regular*. A notable exception occurs in the case of certain series obtained by Liouville (*Journal de Mathématiques*, volume 3) from differential equations of order higher than the second, with boundary conditions of a particular type. It is readily seen that these conditions are irregular, and consequently the question of the convergence of the series, which was left open by Liouville, also falls outside the scope of Birkhoff's general treatment. Dr. Jackson shows that the difference between this case and those usually studied is an essential one; expansions which would be uniformly convergent, if the analogy of the ordinary theory were sustained, are found actually to diverge with great rapidity. The difference is explained by the fact that the characteristic functions, instead of being essentially of the nature of trigonometric functions, as in the familiar cases, involve real exponential factors which fundamentally affect their properties. The analysis is applicable not only to Liouville's boundary conditions, but also to a considerably more general class of conditions in which these are included.

9. Segre has pointed out that on a surface (two-spread) defined by solutions of a partial differential equation of the second order in any space of higher dimensionality than three, there exists one and only one conjugate net of curves, in the sense that the tangents to curves of one family of the net, at the points where these curves meet a fixed curve of the other family, form a developable surface. It is well known also that with every one-parameter family of curves on a surface in three-space may be associated a unique conjugate family. Dr. Green's paper has two purposes in view: first, to study a surface in a space of n dimensions referred to a conjugate net of parameter curves; and second, to provide a substitute for conjugacy on a surface when $n > 3$, in order to be able to associate with a one-parameter family of curves on such a surface a second family uniquely determined by the first. The quasi-conjugacy defined in the paper is a reciprocal relation, and Segre's unique conjugate net plays the same rôle as does the asymptotic net on a surface in ordinary space, in the sense that each family of the conjugate net is quasi-

conjugate to itself. The analysis of the paper consists in a study not of a single partial differential equation of the form

$$Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0,$$

but of a completely integrable system of two equations of which the above is one. Wilczynski's methods are used in obtaining the invariants and covariants of the configurations considered.

10. The non-linear integral equations discussed in Miss Seely's paper are of the type susceptible of treatment by Volterra's algebra of permutable and non-permutable functions. Volterra has proved that the binomial quadratic

$$\int_a^b \xi(s, \tau)\xi(\tau, t)d\tau + 2\xi(s, t) + \lambda k(s, t) = 0$$

is satisfied by a certain power series in λ within its circle of convergence, the coefficients being symbolic powers of the known kernel $k(st)$ except for constant factors. Miss Seely proves that the equation is also satisfied by the analytic prolongation of this series at every non-singular point in the λ plane, and shows that the singular points are identical with the characteristic constants of the kernel $k(st)$ as given by the Fredholm theory. If λ_h is a pole of order $m > 1$ of the Fredholm resolvent kernel $k(st, \lambda)$, it is an infinity of order $(2m - 3)/2$ of the solution of the binomial quadratic; if $m = 1$, it is a branch point of first order. An expression is given for the solution valid at every point in its star. Extension is made to binomial integral equations of nth degree, and to the general quadratic and cubic, and some discussion is given of the question of the existence of solutions at the singular points.

11. President Van Vleck's retiring address will appear in full in the BULLETIN for April.

12. Let a simple closed surface in space of three dimensions, which is made up of a finite number of pieces of analytic surfaces meeting one another along a finite number of analytic curves, be denoted as an analytic polyhedron. The corresponding $(n - 1)$-dimensional manifold in space of n dimensions shall be called an analytic hyperpolyhedron. Professor

Osgood proves the theorem that such a manifold divides the space in which it lies into an interior and an exterior region. The method consists in slicing by a family of parallel planes or hyperplanes and considering the regions into which one of them is divided.

13. Professor Birkhoff's first paper contains an extension of the methods of an earlier paper* to the treatment of certain classes of functions of several complex variables, and will appear in the *Acta Mathematica*.

14. In this paper Professor Birkhoff carries further the principle of "reducibility of maps" employed earlier by him,† and thus limits still further the form of "completely reduced" maps. The importance of the notion of reducibility is that it appears probably to afford an adequate although extremely complex method for the solution of the four-color problem.

This paper will be offered to the *American Journal of Mathematics*.

15. The paper of Professors Snyder and Sharpe has for its purpose the establishment of two theorems:

Theorem I. The quartic subjected to the single condition of passing through a non-hyperelliptic sextic curve of genus three is invariant under an infinite discontinuous group of birational transformations.

Theorem II. The transformations of the infinite discontinuous group under which the most general quartic surface passing through a sextic curve of genus two remains invariant can be expressed in terms of cremonian transformations.

In connection with the first theorem the equation of the surface is derived and the equations of the transformations are determined; it is shown that the transformations are cremonian and non-involutorial and that no transformations exist other than those obtained. It is believed that this surface is the first illustration of one which possesses an infinite discontinuous group, but contains neither a pencil of elliptic curves nor a net of hyperelliptic curves of genus two. The

* "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations," *Proceedings of the American Academy of Arts and Sciences*, vol. 49 (1913), pp. 521–568.

† *American Journal of Mathematics*, vol. 35 (1913), pp. 115–128.

equation of the surface (Fano surface) mentioned in the second theorem is found, and also the equations of two involutorial space transformations, which generate the infinite group.

16. By extending some methods due to Kummer (Berlin *Sitzungsberichte*, 1857) Mr. Vandiver is enabled to set up a relation from which the criteria of Kummer (Mirimanoff, *Crelle*, volume 128), Legendre, and Furtwängler (*Wiener Sitzungsberichte*, 1912, page 589) in reference to Fermat's last theorem may be immediately deduced.

17. In the *Mathematische Annalen*, volume 22 (1883), Stéphanos establishes a one-to-one correspondence between the projectivities of the binary domain and the points of a projective S_3. In Professor Coolidge's paper these formulas are adapted to establishing a one-to-one correspondence between the points of a complex projective S_3 and the real circular transformations of the Gauss plane. The correspondence is used on the one hand to give real representation of various loci in complex space, and on the other to discover new theorems concerning circular transformations.

18. The useful theorem that every subgroup of a group of order p^m, p being any prime number, is invariant under operators of the group which are not contained in this subgroup is commonly supposed to be due to G. Frobenius and to W. Burnside. Cf. "Encyclopédie des Sciences mathématiques" tome I, volume 1, page 608; W. Burnside, "Theory of Groups of Finite Order," 1911, page 122. As a matter of fact, this theorem is evidently a special case of a theorem which was proved eleven years earlier by A. Capelli, and which may be stated as follows: A necessary and sufficient condition that every subgroup of a group G is transformed into itself by at least one operator which is not contained in this subgroup is that G contains no more than one Sylow subgroup of any given order.*

This theorem is clearly equivalent to the theorem that a necessary and sufficient condition that every subgroup of a group G appears in at least one of its possible series of com-

*A. Capelli, *Atti R. Accademia dei Lincei, Memorie*, (3), vol. 19 (1884), p. 272.

position is that G is the direct product of its Sylow subgroups; which the second edition of Pascal's Repertorium der höheren Mathematik, volume 1, 1910, page 199, accredits to W. Burnside. The object of Professor Miller's paper is to direct attention to this and to other theorems proved by A. Capelli, and to extend one of these theorems. In particular, the operators of any composite group G are arranged in the usual rectangular form with respect to an invariant subgroup H as the first row, and the operators of the first column are so selected as to generate the smallest possible group. The cross-cut of this group and H cannot contain more than one operator, or more than one subgroup, in any complete set of conjugates when this is also a complete set of conjugates under G. In particular, this cross-cut is the direct product of its Sylow subgroups. The smallest group above mentioned must be G itself whenever G is cyclic and H does not include any Sylow subgroup of G.

19. An ordinary differential element in the plane may be represented by a point and a parabolic curve of the nth order passing through it. An infinite power series in x, convergent or divergent, Professor Kasner regards as defining a differential element of infinite order. This may be pictured most concretely, in general, by the broken line running from the given point to the successive centers of curvature. If further we allow series with fractional exponents, we obtain, in addition to regular elements, new types of irregular elements. According to the nature of the coefficients in the series, the element may be real or imaginary. In earlier papers the author has classified elements with respect to the conformal group. The present paper gives the classification of all elements (real or imaginary, regular or irregular, convergent or divergent) in metric geometry.

20. Professor Cole obtains a set of very simple resolvent equations for the algebraic solution of quintics of the form $x^5 + \alpha x + \beta = 0$ when such solution is possible. This paper will appear in a later number of the BULLETIN.

21. In order to express the variation of a function depending continuously on all the values of another function between a and b, in terms of the integral of the derivative of the de-

pendent function with respect to the functional argument multiplied by the variation of that argument, Volterra finds it necessary to make use of four postulates, two of which are statements of uniform conditions. In the present paper Professor Evans seeks to reduce the number of these assumptions. He proves that if the functional derivative exists for every point of a continuous curve $\varphi(x)$ between $x = \xi_1$ and $x = \xi_2$, and takes the values α at ξ_1 and β at ξ_2, then between ξ_1 and ξ_2 it takes all the values between α and β. If the functional derivative of the function F exists for all values of ξ between $x = a$ and $x = b$, and for all functions φ in a certain region, being continuous with respect to φ in that region, and if F vanishes for φ_1 and φ_2, $\varphi_2 - \varphi_1$ being everywhere a positive quantity, then the derivative of F vanishes at some point of one of the curves of the pencil $(\varphi_1\varphi_2)$. If the derivative is continuous in regard to its functional argument in the neighborhood of $\varphi(x)$, and finite, and integrable along $\varphi(x)$ in regard to its other argument ξ from a to b, then the first variation may be written in the desired form. Moreover if in addition, the derivative is continuous in regard to ξ, so that two of Volterra's postulates are established, the other two follow.

22. Professor Huntington's paper contains a tentative set of postulates for the composition and resolution of forces, expressed in terms of the following fundamental concepts: (1) a class of elements $a, b, c, \cdots$, to be called forces, or vectors; (2) a relation $a \uparrow b$, to be called like-directedness; a relation $a \equiv b$, to be called congruence; and a rule of combination $a \oplus b = c$, to be called vector addition.

23. The infinite system of equations considered by Professor Moulton may be written in the form

$$x_1 = f_1(x_1, x_2, \cdots; m) = f_1^{(0)} + f_1^{(2)} + \cdots,$$
$$x_2 = f_2(x_1, x_2, \cdots; m) = f_2^{(0)} + f_2^{(2)} + \cdots,$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

where m is an arbitrary parameter, $x_1, x_2, \cdots$ an infinite system of dependent variables, $f_i^{(j)}$ the totality of terms of f_i which are homogeneous of degree j in $x_1, x_2, \cdots$. One set of hypotheses under which the conclusion follows is:

(*a*) The $f_i^{(0)}$ vanish for $m = 0$.
(*b*) The $f_i^{(0)}$ and the coefficients of the $f_i^{(j)}$ are regular analytic functions of m for $|m| \leqq \rho$.
(*c*) There exists a finite constant M_i and a series $c_1x_1+c_2x_2 + \cdots = s$, where the c_i are regular functions of m for $|m| \leqq \rho$, such that s converges if $|m| < \rho$, $|x_i| < r_i$ and $M_i s^j$ dominates $f_i^{(j)}$.
(*d*) $M_i \leqq r_i$ if $|m| \leqq \rho' \leqq \rho$, $|x_i| \leqq r_i' \leqq r_i$.

Under these conditions there exist series of the form

$$x_i = a_i^{(1)}m + a_i^{(2)}m^2 + \cdots$$

which satisfy the original equations identically in m and converge if $|m| < \sigma > 0$.

By an application of these results, suitably modified, the expansions as power series in m of the Fourier coefficients of the expressions for the coordinates in Hill's variational orbit can be shown to converge for $|m|$ sufficiently small. This ocmpletes the proof of the convergence, in sufficiently restricted domains, of all the series Hill employed in his lunar theory.

24. The location of the real roots of the function $P(x)$ defined by

$$P(x) \equiv \frac{\Sigma(-1)^s}{s!}\frac{1}{x+s}$$

has been studied by Bouquet who found that in each of the intervals $-2n < x < -2n+\frac{1}{2}$, $-2n+\frac{1}{2} < x < -2n+1$ there lies *at least* one root, while outside these intervals there are no real roots. Professor Haskins shows by an application of the theorem of Budan-Fourier that in each of these intervals there is *exactly* one root.

25. If $x_i = x_i(s)$, $(i = 1, 2, 3)$, is the parametric representation (in terms of the arc) of an arbitrary but fixed geodesic of a developable surface y, then, as is well known,

$$y_i = x_i - a\left(\frac{R}{T}\alpha_i - \gamma_i\right) \qquad (i = 1, 2, 3),$$

where α and γ are the directions of the tangent and binormal at the general point of the curve x, R and T are the radii of curvature and torsion at this point, and a is a parameter. Dr.

Graustein finds as the finite equation of the geodesics on y

$$\left(c_2 + \frac{R}{T}c_3\right)a - (c_1 + c_3 s) = 0,$$

where c_1, c_2, c_3 are arbitrary constants; and as the finite equation of the geodesic circles on y

$$\left(\frac{a}{r} - k_1\right)^2 + \left(\frac{1}{r}\left(s - a\frac{R}{T}\right) - k_2\right)^2 = 1,$$

where r is the radius of geodesic curvature of the general circle and k_1, k_2 are arbitrary constants. These equations afford simple deductions of facts already known and lead to a number of results which seem to be new.

26. Dr. Barrow considers the subject of oriented circles in three dimensions. He makes use of the Plücker coordinates of a circle which are formed from the pentaspherical coordinates of any two spheres intersecting in the circle; and accomplishes the orientation by means of a redundant coordinate defined by a quadratic relation as a two-valued function of the Plücker coordinates. A geometric interpretation for a certain invariant function of the coordinates of two oriented circles is given; and a few theorems are proved about systems of oriented circles whose coordinates satisfy one or more linear relations.

27. The transformation considered by Professor Maclay consists in replacing the exponents of a polynomial by their positive residues, in particular their least positive residues, relative to an arbitrary modulus, and forming the difference of the two polynomials. The effect of successive applications of the transformation and the relations of the resultant polynomials are studied.

28. Mr. Alexander's paper contains a simple method for transforming any algebraic surface birationally into a surface without singularities lying in a space of sufficiently many dimensions. By a direct generalization, the device may be used to resolve the singularities of any n dimensional manifold.

Let $f(x_0, x_1, x_2, x_3) = 0$ be an algebraic surface. The transformation

$$x_i = \frac{\partial f}{\partial x_i} \qquad (i = 0, 1, 2, 3) \tag{1}$$

transforms f into its dual $F(x_0, x_1, x_2, x_3) = 0$. The transformation

$$\xi_{ij} = x_i x_j, \tag{2}$$

together with (1), transforms f into a surface ϕ in 15-dimensional space. It will be found that every singular point p of f which is not a uniplanar point is transformed into a curve l of ϕ and that the multiplicity of each point of l is lower than that of p. A uniplanar point may be simplified by a transformation (2′) analogous to (2). By means of a finite number of transformations (2) or (2′) followed in each case by a projection of the surface back into 3-space, all the singularities may be resolved.

29. Letting $u_0 + u_1 + \cdots + u_n + \cdots$ be any series, de la Vallée-Poussin (*Bulletin de l'Académie Belgique*, 1908) introduced the expressions

$$V_n = \sum_{v=0}^{n} \frac{n!n!}{(n-v)!(n+v)!} u_v,$$

and the original series is summable when $\lim_{n=\infty} V_n$ exists. In the present paper, Professor Gronwall forms the generating identity of these sums, viz., writing $\xi = 4x/(1+x)^2$,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1+x} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!n!} V_n \xi^n.$$

From this identity, a generalization of Abel's theorem for power series is obtained; it is shown that any series summable by Cesàro's method is also summable by the present method with the same sum; and various other properties of the sums V_n are proved. The theory may be generalized to the sums obtained from the fundamental identity by making

$$\xi = 1 - \left(\frac{1-x}{1+x}\right)^{2m}.$$

30. Professor Gronwall gives a direct proof of the existence and uniqueness of a bounded solution of the integral equation

$$\varphi(x, y) = f(x, y) + \int_0^x A(x, y, s)\varphi(s, y)ds + \int_0^y B(x, y, t)\varphi(x, t)dt + \int_0^x ds \int_0^y C(x, y, s, t)\varphi(s, t)dt,$$

when f, A, B and C are bounded and have their discontinuities regularly distributed in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. This proof avoids the usual decomposition of the rectangle (a, b) into smaller rectangles. An application to partial differential equations of the hyperbolic type is given.

31. If the power series $f(x) = x + a_2x^2 + \cdots + a_nx^n + \cdots$ is such that $z = f(x)$ defines the conformal representation of the unit circle in the x-plane on a simple region in the z-plane, then, as Koebe has shown, there exist upper and lower boundaries for $|f(x)|$ and $|f'(x)|$ when $|x| = r < 1$, these boundaries being independent of the coefficients of $f(x)$.

Professor Gronwall obtains the following approximations to these boundaries:

$$\frac{2}{9}r < |f(x)| < \frac{3r}{(1-r)^2}, \quad \frac{1}{5} \cdot \frac{1-r}{(1+r)^3} < |f'(x)| < 5 \cdot \frac{1+r}{(1-r)^3}.$$

These approximations are of the correct order of magnitude, as is shown by the example $f(x) = x/(1-x)^2$.

F. N. Cole,
Secretary.

THE WINTER MEETING OF THE SOCIETY AT CHICAGO.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Monday and Tuesday, December 28–29, 1914, it being the third regular meeting of the Society at Chicago. Eighty persons were in attendance upon the sessions, including the following sixty-four members of the Society:

Professor R. P. Baker, Professor W. H. Bates, Mr. William Betz, Professor G. A. Bliss, Professor Henry Blumberg, Professor Daniel Buchanan, Professor H. E. Buchanan, Dr. Josephine Burns, Professor W. H. Bussey, Professor W. D. Cairns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. H. Clarke, Professor H. E. Cobb, Professor D. R. Curtiss, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor L. W. Dowling, Professor Arnold Dresden, Professor Arnold Emch, Professor A. B. Frizell, Dr. M. G. Gaba,

Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. L. A. Hopkins, Professor A. M. Kenyon, Dr. H. R. Kingston, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor H. W. March, Professor J. L. Markley, Dr. T. E. Mason, Professor G. A. Miller, Professor U. G. Mitchell, Professor E. H. Moore, Professor E. J. Moulton, Professor L. C. Plant, Mr. V. C. Poor, Professor S. W. Reaves, Professor W. J. Risley, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor R. B. Stone, Professor E. B. Stouffer, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor Mary E. Wells, Professor Marion B. White, Professor E. J. Wilczynski, Professor K. P. Williams, Professor R. E. Wilson, Mr. C. H. Yeaton, Professor J. W. A. Young, Professor W. A. Zehring, Professor Alexander Ziwet.

Professor E. B. Van Vleck, President of the Society, presided at the session on Monday afternoon. Professor E. J. Wilczynski, chairman of the Section and Vice-President of the Society, presided at all the other sessions, except when relieved by Professor C. S. Slichter during part of the session on Monday morning.

At the business meeting of the Chicago Section the report of the committee of the Section appointed last April to consider the relation of the Society to the field now covered by the *American Mathematical Monthly* was received and ordered spread upon the minutes. In accordance with the recommendation of this committee, it was then voted that the Chicago Section request the Council of the Society to appoint a committee to consider and report concerning possible relations of the Society to the field now covered by the *American Mathematical Monthly*. In transmitting this request to the Council the Secretary of the Section was instructed to send with it a copy of the report of the committee of the Section.

The following officers of the Chicago Section were elected: Chairman, Professor E. J. Wilczynski; Secretary, Professor H. E. Slaught; and third member of the program committee, Professor R. D. Carmichael. Professor Wilczynski was requested by the Section to deliver an address upon the completion of his second term of office as chairman.

The usual social gathering and dinner on Monday evening

was attended by sixty members, this being an unusually large number.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "On some general theorems concerning analytic curves."

(2) Professor A. F. CARPENTER: "Ruled surfaces with plane flecnode curves."

(3) Professor E. J. WILCZYNSKI: "The general theory of congruences."

(4) Professor A. B. FRIZELL: "A non-enumerable well-ordered set of infinite permutations."

(5) Professor HENRY BLUMBERG: "On a generalization of certain theorems relating to discontinuous functions."

(6) Professor DANIEL BUCHANAN: "A new isosceles triangle solution of the three-body problem."

(7) Professor DANIEL BUCHANAN: "Oscillations near an isosceles triangle solution as the finite bodies become unequal."

(8) Mr. L. L. STEIMLEY: "On the general solution and so-called special solutions of linear homogeneous partial differential equations."

(9) Mr. W. L. HART: "A method of successive approximations for the determination of implicit functions."

(10) Professor ARNOLD DRESDEN: "On the second derivative of the extremal integral for a general class of problems."

(11) Professor G. A. BLISS: "A note on functions of lines."

(12) Professors G. A. BLISS and A. L. UNDERHILL: "Shortest lines between curves and surfaces."

(13) Dr. G. R. CLEMENTS: "Singular point transformations in two complex variables. Supplementary note."

(14) Professor HENRY BLUMBERG: "On the factorization of certain polynomials and certain linear homogeneous differential expressions. Second communication."

(15) Dr. G. R. CLEMENTS: "The resultant of a system of polynomials."

(16) Mr. A. R. SCHWEITZER: "On the formal properties of functional equations."

(17) Professor L. E. DICKSON: "On modular cubic surfaces."

(18) Professor G. A. MILLER: "A new proof of Sylow's theorem."

(19) Professor G. A. MILLER: "Note on the potential and the antipotential group of a given group."

(20) Professor J. B. SHAW: "On parastrophic algebras."

(21) Mr. C. H. FORSYTH: "A general formula for the valuation of bonds."

(22) Dr. H. R. KINGSTON: "Metric differential properties of nets of plane curves."

Mr. Steimley was introduced by Professor R. D. Carmichael, Mr. Hart by Professor F. R. Moulton, and Mr. Forsyth by Professor J. W. Glover. The papers of Dr. Clements and Mr. Schweitzer were read by title.

Abstracts of the papers follow, the numbers corresponding to those of the titles above.

1. In an article which will appear in the *Rendiconti del Circolo Matematico di Palermo* it is proved analytically that every closed analytic curve without cusps and multiple points has an even number of points of inflexion and admits of an even number of tangents parallel to any given direction. The proofs of these theorems are based upon the co-periodicity of the parametric representation of such curves and of their derivatives.

These results may be extended to any closed analytic curve by properly extending the definition of inflexions and tangents and by taking proper care of multiplicities. Among such closed curves may be included those that result from the former by projective transformations.

A purely intuitive proof of the theorem on inflexions was given by von Staudt in his Geometrie der Lage. In the present paper Professor Emch extends his former proofs to the general cases and discusses von Staudt's method.

2. In the general theory of ruled surfaces as developed by Wilczynski, many important theorems are obtained by taking for directrix curves the two branches of the flecnode curve. In Professor Carpenter's paper the condition that these curves be plane leads to the theorem: If the two branches of the flecnode curve of a ruled surface are distinct plane curves, then the two branches of the complex curve are plane and the four planes of these curves form a harmonic pencil of planes in which the planes of the two branches of the flecnode curve are separated by the planes of the two branches of the complex curve.

3. Professor Wilczynski's paper is concerned with the projective differential properties of congruences of rays.

By introducing a number of new congruences covariantly related to the given one, many new theorems present themselves and most of the familiar ones appear in a new light. Incidentally a new geometric interpretation is found for the condition that the Laplace-Darboux invariants of a conjugate system be equal. There is also found the geometrical significance of the systems which Bianchi calls isothermally conjugate.

4. In a previous paper (Chicago, April, 1914) Professor Frizell has called attention to a non-denumerable well-ordered set of finite permutations. The present note applies this result to obtain a non-enumerable well-ordered set of infinite permutations of the series of natural numbers. This is done by arranging the natural numbers in series higher than ω.

5. In his first paper, Professor Blumberg shows that almost all of the known general theorems on real, discontinuous functions involving the oscillation function (which is associated with every given function, whether discontinuous or continuous) hold for the case where the range of the independent variable is an abstract aggregate $\mathfrak{S}$ subject to two simple conditions. The first demands for every element s of $\mathfrak{S}$ the existence of a sub-aggregate $\mathfrak{M}$ of $\mathfrak{S}$ having the relation R to s, R being an undefined relation. The second says that if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two "neighborhoods" (the term "neighborhood" having been defined in terms of the undefined relation R) of an element s of $\mathfrak{S}$, there exists a common sub-aggregate $\mathfrak{N}$ of $\mathfrak{N}_1$ and of $\mathfrak{N}_2$ that is a "neighborhood" of s. In particular, for example, the Sierpinski theorem holds. Certain general theorems on discontinuous functions communicated by the author in former papers are similarly generalized.

6. The isosceles-triangle solutions of the three body problem are the periodic solutions in which two of the bodies have equal mass, and the third body moves so that its distances from the other two bodies are always equal. The third body, then, moves in a straight line which passes through the center of gravity of the system and is perpendicular to the plane of the initial motion of the equal masses. In a paper presented to the Society at Chicago, April, 1911, Professor Buchanan discussed the periodic solutions in which the two

equal masses move in ellipses and the third body is infinitesimal. Two periodic solutions were shown to exist. They are expansible as power series in a certain parameter λ, which is introduced through the factor representing the mass of the finite bodies, and the coefficients are power series in e, the eccentricity of the ellipses. The period T is a function of λ but not of e. In the present paper two periodic solutions, having period T, are shown to exist when the third body becomes finite. They are expansible as power series in a parameter representing the mass of the third body, and the coefficients are power series in λ and e similar to those appearing in the previous paper.

7. Starting with the simplest case of the isosceles-triangle solutions in which the third body is infinitesimal and the finite bodies move in circles, Professor Buchanan in his second paper deals with the oscillations near this solution as the finite masses become unequal. If the finite bodies have masses $m + \mu$ and $m - \mu$, where the unit of mass is chosen so that $m = \frac{1}{2}$, then the solutions are expansible as power series in μ. The coefficients are power series in another parameter which represents the initial projection of the infinitesimal body from the plane of motion of the finite bodies where $\mu = 0$. Two solutions are shown to exist and they are expansible as power series in $\mu^{\frac{1}{2}}$ or μ according as the purely imaginary characteristic exponents arising from the equations of variation are imaginary integers or not.

8. It is known that the customary classification of the integrals of linear partial differential equations is incomplete even in the case where the coefficients are single valued and analytic in a given domain. For this latter case Mr. Steimley gives a complete classification of all possible integrals and exhibits a solution which is general in the sense that it includes all possible solutions (even the so-called special solutions).

9. Let there be given the system of equations

$$f_i(x_j; \mu) = 0 \qquad (i, j = 1, 2, \cdots, n), \tag{1}$$

in the field of reals, with the solution point $(a_j; \mu_0)$ at which the Jacobian, $\partial(f_1, \cdots, f_n)/\partial(x_1, \cdots, x_n)$, is different from zero. The current methods by which the fundamental

theorem of implicit function theory is proved for system (1) all possess the fault that no practical scheme is given for the computation of the values of the $x_j(\mu)$ except for μ sufficiently near to μ_0. In Mr. Hart's paper there is given a method of successive approximations which furnishes a practical scheme for computing the values of the $x_j(\mu)$ over a range in general much more extended than that obtained in the usual proofs.

10. In a paper published in volume 9 of the *Transactions* of the Society, Professor Dresden treated problems of the calculus of variations by reducing them to problems in ordinary maxima and minima by means of the "extremal integral." This necessitated securing formulas for the second derivatives of this integral; these were obtained in the paper mentioned above and they are quoted on page 312 of Bolza's Vorlesungen.

It is the purpose of the present paper to secure similar formulas for more general problems of the calculus of variations, viz., such as involve integrals whose integrand contains n unknown functions and their first derivatives, in non-parametric form as well as in parametric form. In the former case these formulas are obtained without great difficulty. In the latter case, however, it has seemed to the author that a reduction of the second variation was necessary. This is done by means of a reduction of differential forms discussed in a paper presented to the Society some time ago and published in the *Annals of Mathematics* for March, 1912.

It is intended to use these formulas for the treatment of the maxima and minima of integrals of the type referred to above when one or both end points are variable, as well as when broken extremals are admitted. Furthermore the cases are to be taken up in which the unknown functions are also to satisfy a system of differential equations.

11. One of the most important applications of the theory of functions of lines is in the calculus of variations, where the integrals considered are all either functions of this sort or else quite natural generalizations of them. Unfortunately the usual definitions of continuity and the derivative of a function of a line do not apply to such integrals. In the paper of Professor Bliss the character of the continuity of an integral of the calculus of variations is discussed and the inadequacy of the definition of the derivative is shown.

Attention is called to the modifications which make these notions applicable at once to such line functions.

12. If an arc E in space joins two surfaces S and S' and is shorter than any other arc in its neighborhood joining S with S', then E must in the first place be a straight line, and in the second place it must cut the two surfaces orthogonally. These conditions are necessary but not sufficient to insure the minimizing property. The further relations which must be satisfied involve the curvatures of the surfaces at their intersections with the line E. In the paper of Professors Underhill and Bliss these relations are deduced, and analogous ones are found for the cases where the end points of E lie upon two curves, or on a surface and a curve. The results are of interest because of the preliminary information which they give with respect to the more general problem of determining the minimizing arcs, with end points variable on curves or surfaces, for an integral of the form

$$\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx.$$

13. In the *Annals of Mathematics* for September, 1913, Dr. Clements discussed the transformation

$$T: \qquad x = f(u, v), \qquad y = \varphi(u, v)$$

in the following cases:

(*a*) $f(u, v)$ and $\varphi(u, v)$ denote functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0, v = 0$;

(*b*) $$f(0, 0) = 0, \qquad \varphi(0, 0) = 0;$$

(*c*) $$J(u, v) \equiv f_u\varphi_v - f_v\varphi_u \not\equiv 0, \qquad J(0, 0) = 0.$$

In that paper the discussion was not complete for the case in which $f(u, v)$ and $\varphi(u, v)$ have a common factor in the point (0, 0) but neither is a factor of the other in this point. In the present note it is shown that for this case the inverse of T exists and is finitely multiple-valued and continuous throughout a complete neighborhood of the point $x = 0$, $y = 0$, with the exception of that point itself, which explodes into a locus in the uv-region. This establishes the existence of

functions of two complex variables, finitely multiple valued and continuous except in isolated points.

Any transformation T can always be replaced either by a transformation of the form $x = u,\ y = v^n$ (n a positive integer), a finite number of the form $x = u,\ y = uv$, and of those one-to-one and analytic both ways; or by the preceding combined with a transformation T for which $f(u, v)$ and $\varphi(u, v)$ have no common factor.

14. In his second paper, Professor Blumberg gives criteria for the resolvability and, in particular, the irreducibility of certain polynomials, linear homogeneous differential expressions, and other expressions. The methods used are the simplest heretofore employed in similar investigations; the results are the most general heretofore obtained and, largely on account of the simplicity of the methods, generalization is made easy. The most general theorem communicated by the author at the November, 1914, meeting of the Southwestern Section is a special case of a special case of one of the theorems in the present paper. To indicate the character of the results one of the theorems obtained and several special cases of it are given below. Let

$$P = a_0(x)\frac{d^p y}{dx^p} + a_1(x)\frac{d^{p-1}y}{dx^{p-1}} + \cdots + a_p(x)y$$

be a linear homogeneous differential expression belonging to (i. e., whose coefficients belong to) the domain $\mathfrak{R}$ of rational functions; let d_ν ($\nu = 0, 1, \cdots, p$) represent the degree of a_ν; and let P be decomposable into the symbolic product RS, where R and S are linear homogeneous differential expressions belonging to $\mathfrak{R}$ and of the rth and sth orders respectively. The following theorem holds: If for a fixed κ (> 0)

$$\frac{d_\kappa - d_0}{\kappa} \geqq \frac{d_\nu - d_0}{\nu} \qquad (\nu = 1, 2, \cdots, p)$$

and

$$d_\kappa - d_0 \geqq 0,$$

at least one of the following $n - \kappa + 1$ congruences holds:

$$d_\kappa - d_0 \equiv 0 \left(\mod \frac{\kappa}{[\kappa, \rho]}\right),$$

where ρ takes the successive values $n-r, n-r+1, \cdots, \kappa-r$ (or $n-s, n-s+1, \cdots, k-s$), and $[\kappa, \rho]$ = G.C.D. of κ and ρ. A special case of this theorem is the following: If

$$\frac{d_\kappa - d_0}{\kappa} \geqq \frac{d_\nu - d_0}{\nu} \quad (\nu = 1, 2, \cdots, p),$$

$d_\kappa - d_0 \geqq 0$ and $[d_\kappa - d_0, \kappa] = 1$ (i. e., $d_\kappa - d_0$ and κ are relatively prime), either $r \geqq \kappa$ or $s \geqq \kappa$. From this special case follows the theorem: If

$$\frac{d_p - d_0}{p} \geqq \frac{d_\nu - d_0}{\nu} \quad (\nu = 1, 2, \cdots, p),$$

$d_p - d_0 \geqq 0$ and $[d_p - d_0, p] = 1$, P is irreducible.

The above theorems and nearly all others obtained hold for polynomials whose coefficients are rational functions of one or more variables, for linear difference equations, and also for certain cases where the domain of rationality is more general than $\mathfrak{R}$, the only changes required in the formulation of the theorems being the substitution, for example, of "difference" for "differential" in passing to "difference equations," etc.

One of the theorems generalizes the most general criterion for the irreducibility of polynomials heretofore known, a criterion due to Perron.

15. It is well known that in any *numerical* case there exist $N-1$ rational integral functions of the coefficients of a system of polynomials

$$f_n(x) \equiv A_{n0}x^{m_n} + \cdots + A_{nm_n} \quad (n = 1, \cdots, N)$$

whose simultaneous vanishing is the necessary and sufficient condition that these N polynomials shall have a common factor of the first or higher degree in x. Further, if $N = 2$, this resultant can be written formally in terms of the coefficients. Dr. Clements points out that when $N > 2$, if the coefficients are real, there exist $N-1$ rational integral functions, expressible as formulas in the coefficients and hence in their definition independent of the numerical value of these coefficients, whose simultaneous vanishing is the necessary and sufficient condition that $f_1, \cdots, f_N$ shall have a common factor containing x. Further, infinitely many such

sets of resultants can be found. A very simple set of resultants is exhibited for the case $N = 3$, $m_3 = 2$.

16. In the first part of Mr. Schweitzer's paper a set of postulates is given for a *field* in which the explicit assumption of existential properties is avoided. These postulates are stated in terms of four indefinables, $f(x, y)$, $\phi(x, y)$, $\phi_1(x, y)$, $f_1(x, y)$, and are obtained essentially by giving abstract interpretations to functional relations discussed in a previous paper.*

In the second part of the paper the following theorems are proved:

I. If $f\{\beta f(x, z), f(x, y)\} = \alpha\beta f(y, z)$, then there exists $\chi(x)$ such that $\chi f(x, y) = c_0\chi(x) - c_0\chi(y)$, $\chi\alpha(x) = c_0\chi(x)$, and $\chi\beta(x) = \chi(x) + c_1$, where c_0 and c_1 are constants.

II. If

$$f\{f(x, z), f(y, z)\} = \alpha f(x, y)$$

and

$$f_1\{\alpha f_1(x, z), f_1(y, z)\} = \alpha f_1(x, y),$$

then there exists $\chi(x)$ such that $\chi f(x, y) = c_0\chi(x) - c_0\chi(y)$, $\chi\alpha(x) = c_0\chi(x)$ and $\chi f_1(x, y) = c_1\chi(x)/\chi(y)$, that is, $\alpha^{-1}f(x, y)$ and $f_1(x, y)$ are quasi-transitive.

III. If $f\{f_1(x, z), f_1(y, z)\} = f_1(x, y)$ and $f_1\{f(x, z), f(y, z)\} = f(x, y)$, then $f(x, y)$ and $f_1(x, y)$ are identical.

17. The method employed in the paper by Professor Dickson for the investigation of surfaces of any order in modular space is based upon the projective classification of all sets of real points (i. e., points with integral coordinates). For the case of modulus 2, there are exactly $n - 1$ projectively distinct types of sets of n real points when $1 < n < 8$. Since there are just 15 real points, the number of types of sets of n points is the same as the number of types of sets of $15 - n$ points. Hence a case $n \geqq 8$ reduces to a case $n < 8$. To investigate the projectively distinct types of surfaces of a given order with integral coefficients taken modulo 2, we treat as a separate case those surfaces the only real points on which are those of a given canonical set of n real points ($n = 0, 1, \cdots, 15$), so that we have 15 linear relations between the coefficients of the

* Cf. BULLETIN, October, 1914, p. 25, (6); p. 26, (11); p. 28, the relation following (22).

equation of the surface. The transformations available for the further specialization of the coefficients are the automorphs of the set of n points.

For a cubic surface modulo 2, it is shown that the number of real points n on the surface is always odd. For $n = 15$, the surface is equivalent to $xy(x + y) = 0$ or to $xy(x + y) = zw(z + w)$, the latter having no singular point and having 27 straight lines, of which 15 are real. For $n = 1$, there is a single type of surface; it has no singular point and no real line. For $n = 13$, one of the two types has a singular point, while the other is equivalent to

$$xy(x + y) = w(w + z)(y + z),$$

9 of whose 27 lines are real. On one of the five types for $n = 3$, just one of the 27 lines is real. For $n = 11$, there is a type 5 of whose 27 lines are real. Thus there are types of cubic surfaces modulo 2 having exactly 0, 1, 5, 9 or 15 real lines. In the algebraic theory, the only possible numbers for non-singular cubic surfaces are 3, 7, 15, 27. The paper will appear in the *Annals of Mathematics*.

18. Professor Miller bases his new proof of Sylow's theorem on the following theorem: The number of the substitutions of degree p^β and of order p in the symmetric group of degree n is prime to p whenever p^β is the highest power of p which does not exceed n. By means of this elementary theorem Sylow's theorem may be proved as follows: Suppose that the order g of any group G is divisible by p^α but not by $p^{\alpha+1}$. Now represent G as a regular substitution group, and construct all the substitutions on the g letters of G such that these substitutions are of order p and of degree p^β, p^β being the highest power of p which is less than n.

These substitutions are transformed under G into complete sets of conjugates, each set involving more than one. At least one of these sets must involve a number m of substitutions such that m is prime to p, since the total number of these substitutions is prime to p. Hence G contains a subgroup whose order is divisible by p^α. If the order of this subgroup exceeds p^α the given operations may be repeated, and hence we can always reach a subgroup of order p^α, by the given process, provided the order of G is divisible by p^α.

19. Professor Miller's second paper was published in full in the February BULLETIN.

20. The parastrophic algebra is defined with regard to a given algebra by the relation between the multiplication constants

$$\gamma'_{ijk} = \gamma_{ikj} \qquad (i, j, k = 1, \cdots, r),$$

where r is the order of the algebra. The parastrophic algebra is usually not associative when the original algebra is associative. In the case of algebras defined by finite groups the parastrophic algebra takes a very simple form. Professor Shaw's paper discusses the relations between these algebras.

21. In Mr. Forsyth's paper a general formula is derived for the valuation of bonds where the principal of S dollars is to be repaid in $r_1, r_2, \cdots, r_n$ installments, of which the first r_1, the next r_2, etc., are equal, there being *n different* installments. All other similar formulas apply only to cases involving equal installments.

The first installment is to be paid at the end of f_1 years, the next $r_1 - 1$ at intervals of t_1 years, etc., and in general if we designate as "major intervals" those intervals wherein all the installments are equal, there will be r_n installments at intervals of t_n years in the nth major interval, the first installment to be paid at the end of f_n years.

The rate of interest offered is g and the rate to be realized i, both to be paid m times a year.

The premium or discount becomes

$$K = \left[1 - \frac{\sum_1^n a:n}{s}\right]\left(\frac{g-i}{i}\right), \tag{1}$$

where

$$a:n = A_n\{a_m(f_n + r_n t_n) - a_m f_n\}, \qquad A_n = I_n/a_{mt_n}$$

and I_n represents the value of each of the installments in the nth major interval. The a's with subscripts refer to the ordinary annuity certain.

If all the minor intervals are equal ($= t$), formula (1) reduces to

$$K = \left[1 - \frac{\sum_0^n (I_n - I_{n+1})a(n)}{sa_{mt}}\right]\left(\frac{g-i}{i}\right), \tag{2}$$

where

$$a(n) = a_m\left(f_1 + t\sum_1^n r_n\right)$$

and

$$I_0 = I_{n+1} = 0.$$

22. In a memoir entitled "One-parameter families of nets of plane curves" Professor Wilczynski has studied the projective differential properties of nets of plane curves, by means of a completely integrable system of three linear partial differential equations of the second order. In Dr. Kingston's paper some of the metric differential properties of these nets are discussed. If the two linearly independent, non-constant solutions of the partial differential equations

$$y_{uu} = ay_u + by_v, \quad y_{uv} = a'y_u + b'y_v, \quad y_{vv} = a''y_u + b''y_v$$

be taken as the cartesian coordinates of the moving point, it is shown that for a given set of coefficients $a, b, \cdots, b''$, which are functions of the independent variables u and v and are subject to certain integrability conditions, these equations determine a net uniquely except for an affine transformation. If, moreover, we consider the fundamental quantities E, F, and G, which are connected with the length of arc by the relation $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, and satisfy certain conditions, and if we add the condition that these shall have given values, then our original equations along with the given E, F, and G, determine a net uniquely except for a motion and a reflection. The application to orthogonal and isothermal nets gives rise to interesting results. In this paper is laid also the foundation for a detailed study of the metric properties of those nets which are composed of two families of conics.

H. E. SLAUGHT,
Secretary of the Chicago Section.

THE STRUCTURE OF THE ÆTHER.

BY DR. H. BATEMAN.

(Read before the American Mathematical Society, February 28, 1914.)

§ 1. *Introduction.*

In Maxwell's electromagnetic theory of light the broad view is taken that waves of light are manifestations of certain electromagnetic phenomena in the æther which are included in the general class of processes capable of being described by means of the partial differential equations

$$\text{rot } M = -\frac{i}{c}\frac{\partial M}{\partial t}, \quad \text{div } M = 0. \tag{1}$$

In these equations M denotes the complex vector $H + iE$, the vectors E and H represent the electric and magnetic intensities respectively, c is a constant which is usually called the velocity of light, and rot M denotes the complex vector whose components are of type

$$\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

when right-handed rectangular axes are used.

In the extensions of the theory which have been made by H. A. Lorentz* and Sir Joseph Larmor,† it is recognized that the fundamental equations (1) must not be regarded as holding for all real values of the variables x, y, z, t; the exceptional domains or singular points are, moreover, regarded as instrumental in the production of waves or disturbances in the æther.

The hypothesis generally adopted by Lorentz is that there are exceptional domains, which, when viewed at any instant, consist of an enormous aggregate of small closed regions. These domains are supposed to be occupied by electricity

* *Archives néerlandaises*, vol. 25 (1892); *Amsterdam Proceedings* (1902), p. 305.

† *Phil. Trans.* A, vol. 185 (1894), p. 719; vol. 186 (1895), p. 695; Æther and Matter, Cambridge (1900).

whose volume density ρ is generally a function of x, y, z, t. The motion of the electricity is supposed to be specified by a vector function v.

The vector functions E and H are defined for all real values of $x, y\ z, t$ with the aid of the retarded potentials of L. Lorenz. These potentials A and Φ are expressed as triple integrals involving ρ and v and the vectors E, H are derived from them with the aid of the relations

$$(2) \qquad E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial \Phi}{\partial x}, \qquad H = \text{rot } A.$$

When E and H are defined in this way they are found to satisfy equations (1) when the space-time point (x, y, z, t) does not belong to the exceptional domains: for points in the regions occupied by electricity, the modified equations

$$(3) \qquad \text{rot } M = \rho v - \frac{i}{c}\frac{\partial M}{\partial t}, \qquad \text{div } M = i\rho$$

are satisfied.

Starting from these equations Lorentz and Larmor have obtained, by a process of averaging, a set of equations capable of describing the principal optical and electromagnetic properties of a material medium. These more complicated equations are usually called the macroscopic equations, and equations (3) the microscopic equations or fundamental equations of the theory of electrons.

Sir Joseph Larmor has used a more fundamental theory* in which the simple equations (1) are assumed to be the whole scheme and electricity is regarded as consisting of so-called isolated point singularities in the æther, that is, points at which E and H become infinite in a certain manner. A solution of equations (1) which can be regarded as giving an exact specification of an elementary electromagnetic field with a single isolated point singularity of the required type has been obtained by A. Liénard† for the case in which the velocity of the singularity is always less than that of light. It turns out that the electric charge associated with the singularity does not vary with the time. With the aid of this solution, which was originally derived from the retarded

* Æther and Matter, p. 77.
† *L'Eclairage électrique*, t. 16 (1898), pp. 5, 53, 106.

potentials for a volume distribution of electricity, it is possible to reverse matters and effect a transition from equations (1) to equations (3) by a process of averaging or superposition. The exceptional domains within which ρ and ρv are different from zero are now supposed to be occupied by a closely packed aggregate of the point singularities of the superposed elementary electromagnetic fields.

One advantage of this more fundamental theory is that it emphasizes two properties of electricity, namely the invariance of the electric charge and the limitation to which its velocity of motion is subject.* The mathematical discussion of the theory is at present incomplete, partly because some hypothesis is needed which will reveal the structure of the æther and show clearly that electric charges are indeed point singularities in the medium. In this note an hypothesis is considered which fulfils some of the requirements; the mathematical analysis is, however, still incomplete.

§ 2. *Vector Fields with Moving Singular Curves.*

I have shown elsewhere† that there are vector fields satisfying equations (1) in which the vector functions E, H become infinite at points lying on certain moving curves. There is, moreover, at present no evidence that these fields can be built up by superposition from elementary electromagnetic fields of Liénard's type. I propose, therefore, to call them *æthereal fields* so as to distinguish them from the *electromagnetic fields* which possess the property just mentioned. Whether æthereal fields have a physical existence or not is an open question, but I propose to consider the hypothesis that when a large number of æthereal fields are superposed their singular curves indicate the structure of an "æther" which is capable of supporting a certain type of electromagnetic field which is mathematically determined by the properties of the superposed æthereal fields.

According to this view the structure of the æther depends

*This limitation was first clearly pointed out by J. J. Thomson, Recent Researches (1893), p. 21. See also O. Heaviside, *Phil. Mag.*, vol. 27 (1889), p. 324.

† *Philosophical Magazine*, Oct., 1913, Jan., 1914; *Annals of Mathematics*, 1914; The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equations, Cambr. Univ. Press, 1914, Ch. 8. This will be denoted hereafter by *E*.

to some extent upon the type of electromagnetic field which it supports, and vice versa. There must consequently be some definite mathematical relation between the electromagnetic field and the superposed æthereal fields whose singular curves indicate the structure of the æther. We shall now consider a relation which is suitable for the case in which the electromagnetic field is an elementary field of Liénard's type.

§ 3. *Conjugate Fields.*

Two vector fields in which the complex vectors are M, M' respectively are said to be conjugate when the scalar product (MM') is zero. A field which is conjugate to itself is said to be self-conjugate and in this case $(M^2) = 0$.

If the direction of the flow of energy in two fields is the same at each point and both fields are self-conjugate they are also conjugate to one another. To prove this we remark that since the direction of the flow of energy is at right angles to both the electric and magnetic intensities, the vectors E, H, E', H' associated with any point must lie in one plane. Moreover, since both fields are self-conjugate the electric and magnetic intensities in each field are perpendicular, hence we must have $M' = kM$, where k is a complex quantity. It is now clear that the relation $(MM') = 0$ is a consequence of the two relations $(M^2) = 0$, $(M'^2) = 0$. It should be noticed that the present argument fails if the flow of energy takes place in *opposite* directions in the two fields, because then the correct conclusion is that $M' = kM^*$ where

$$M^* = H - iE.$$

Let us now consider an electromagnetic field and a large number of mutually conjugate self-conjugate æthereal fields each of which is conjugate to the electromagnetic field. We shall regard the singular curves of these æthereal fields as forming an *element of the æther* and shall suppose that it is this element which supports the particular electromagnetic field in question. In the example of an element which will be given presently the singular curves meet in a point which is the point singularity of the electromagnetic field supported by the element.

It should be remarked that the condition of conjugacy is invariant when a transformation of variables is made which

leaves equations (1) unaltered in form. Our hypothesis is thus compatible with the modern ideas of relativity. It should also be noticed that when a number of mutually conjugate fields are superposed the term of type $(E^2) - (H^2)$ in the Lagrangian function of the total field is the sum of the terms of this type belonging to the different constituent fields. It is possible then that the condition of conjugacy has a dynamical significance. The fields used in forming an element of the æther can be regarded as analogous to the terms of a Fourier series or the elements of a Fourier integral; each constituent field is supposed to be multiplied by a factor or coefficient, and by giving different values to these coefficients different states of vibration of the fibers of the element may be represented mathematically. With regard to the nature of this vibration or disturbance which can run along a singular curve or fiber the only certain information we are able to give at present is that its orientation may vary with the time and be different for different fibers.

It is by no means certain that the fibers of an element should be regarded as filling the whole of space. If we consider only those æthereal fields whose singular curves lie within a kind of tube or cone having its vertex at the point singularity of the electromagnetic field, it appears likely, judging from some simple cases which have been considered,* that it is possible to choose the coefficients so that when both the electromagnetic and æthereal fields are superposed the vectors E, H in the total field are zero outside the tube. This corresponds to the type of element which has been considered by Sir Joseph Thomson in his theory of the structure of the electric field.†

§ 4. *A Solution of the Fundamental Equations.*

The components of a vector M which satisfies the equations (1) are known to be solutions of the wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{4}$$

The type of singular curve which is typical of an æthereal

* See for instance E, p. 121.

† Recent Researches, Ch. I; *Phil. Mag.*, vol. 19 (1910), p. 301; Ibid., Oct., 1913, Dec., 1913. See also N. R. Campbell, *The New Quarterly*, 1909; D. N. Mallik, *Phil. Mag.*, vol. 26 (1913), p. 144.

field is obtained by considering the case in which the components of M are all wave functions of type $u = \gamma f(\beta)$, where the function f is different for each component but the functions β and γ are the same. The quantities β and γ are supposed to be functions of x, y, z, t; an essential property of β is that it must be complex so that the equation $1/f = 0$ may be equivalent to two real relations connecting x, y, z, t and so generally represent a moving curve. It has been shown that the velocity of this singular curve in a direction at right angles to the tangent at any point is never greater than the velocity of light.*

The problem of finding all solutions of the wave equation having the general form $\gamma f(\beta)$ has not been solved completely, but in the particular case in which this expression can be regarded as included in a more general expression of type $u = \gamma f(\alpha, \beta)$, where f is an arbitrary function and u satisfies the wave equation, the problem can be solved as follows:

It is easily seen that the function f must satisfy the differential equation

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial f}{\partial t}\right)^2 \tag{5}$$

which gives rise to three equations connecting α and β when the coefficients of

$$\left(\frac{\partial f}{\partial \alpha}\right)^2, \quad \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}, \text{ and } \left(\frac{\partial f}{\partial \beta}\right)^2$$

are equated to zero.

Now let x, y, α, β be taken as new independent variables; the preceding equation then takes the form

$$K^{-2}[AX^2 - 2HXY + BY^2] = 0, \tag{6}$$

where

$$A = \left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 - \frac{1}{c^2}, \quad B = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1,$$

$$H = \frac{\partial z}{\partial x}\frac{\partial t}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial t}{\partial y}, \qquad K = \frac{\partial(z, t)}{\partial(\alpha, \beta)},$$

$$X = \frac{\partial(f, z)}{\partial(\alpha, \beta)}, \qquad Y = \frac{\partial(f, t)}{\partial(\alpha, \beta)}.$$

* *E*, p. 123.

Equating to zero the coefficients of

$$\left(\frac{\partial f}{\partial \alpha}\right)^2, \quad \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}, \quad \left(\frac{\partial f}{\partial \beta}\right)^2,$$

we obtain three linear equations connecting A, B, H. The determinant of the coefficients is easily found to be K^{-3}, and this is zero only when there is a relation between α, β, x, y. Ignoring this trivial case, we may conclude from (6) that $A = 0$, $B = 0$, $H = 0$. Now in virtue of the last equation we may write

$$\frac{\partial t}{\partial x} = \lambda \frac{\partial z}{\partial y}, \quad \frac{\partial t}{\partial y} = -\lambda \frac{\partial z}{\partial x},$$

where λ is some function of x, y, α, β. Substituting in the equation $A = 0$, we find that $c^2\lambda^2 + 1 = 0$, hence

$$c\frac{\partial t}{\partial x} = i\frac{\partial z}{\partial y}, \quad c\frac{\partial t}{\partial y} = -i\frac{\partial z}{\partial x}.$$

These equations imply that

$$z - ct = F(x + iy), \quad z + ct = G(x - iy),$$

where F and G are functions which depend also on α and β. Substituting these expressions in the equation $B = 0$, we find that

$$F'(x + iy)G'(x - iy) + 1 = 0.$$

Hence we may put

$$(7) \quad z - ct = \phi + \theta(x + iy), \quad z + ct = \psi - \frac{1}{\theta}(x - iy),$$

where θ, ϕ, ψ are arbitrary functions of α and β. These equations may be used to define α and β as functions of x, y, z, t. It has already been shown* that when α and β are defined by equations of this type a solution of the wave equation (4) may be obtained by writing $u = \gamma f(\alpha, \beta)$ where γ is one of the Jacobians of type $\partial(\alpha, \beta)/\partial(y, z)$. The preceding analysis indicates that all the wave functions of this type may be obtained by the method just described; to complete the proof we must show that the ratio of two suitable values of γ depends only on α and β.

* *Messenger of Mathematics*, March, 1914, p. 164; *E*, p. 134.

Let us suppose that $\gamma f(\alpha, \beta)$ and $\gamma' F(\alpha, \beta)$ satisfy the wave equation, where f and F are arbitrary; then it is easy to deduce that

$$\frac{\partial w}{\partial x}\frac{\partial \alpha}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \alpha}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial \alpha}{\partial z}=\frac{1}{c^2}\frac{\partial w}{\partial t}\frac{\partial \alpha}{\partial t},$$

$$\frac{\partial w}{\partial x}\frac{\partial \beta}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \beta}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial \beta}{\partial z}=\frac{1}{c^2}\frac{\partial w}{\partial t}\frac{\partial \beta}{\partial t},$$

where $w = \gamma'/\gamma$. These equations imply that

$$\frac{\partial w}{\partial x} = P\frac{\partial \alpha}{\partial x}+Q\frac{\partial \beta}{\partial x}, \quad \frac{\partial w}{\partial y} = P\frac{\partial \alpha}{\partial y}+Q\frac{\partial \beta}{\partial y}, \quad \text{etc.};$$

for $w = \alpha$ and $w = \beta$ are evidently particular solutions of the preceding equations. We now have $dw = Pd\alpha + Qd\beta$, whence it follows that w is a function of α and β.

I have shown elsewhere* that if α and β are defined by the equations

$$(8)\qquad \begin{aligned}(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2 &= c^2(t-\tau)^2,\\ l(x-\xi)+m(y-\eta)+n(z-\zeta) &= cp(t-\tau),\end{aligned}$$

where $\xi, \eta, \zeta, \tau, l, m, n, p$ are functions of α and β subject to the relation $l^2 + m^2 + n^2 = p^2$, then wave functions of type $\gamma f(\alpha, \beta)$ exist. It is easily seen that equations (8) are equivalent to (7) if

$$\phi = \zeta - c\tau - \theta(\xi + i\eta), \quad \psi = \zeta + c\tau + \frac{1}{\theta}(\xi - i\eta),$$

$$\theta = -\frac{l - im}{n + p} = \frac{z - \zeta - c(t - \tau)}{x - \xi + i(y - \eta)}.$$

An æthereal field may be specified with the aid of the functions α, β by defining the components of M by equations of type†

$$(9)\qquad M_x = f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i}{c}f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(x, t)}.$$

§ 5. *Model of an Element of the Æther.*

Let us now consider the case when $\tau = \alpha$ and ξ, η, ζ are functions of α only; then as Liénard has shown, the first of

* *E*, Ch. 8.
† *E*, pp. 12, 122.

equations (8) will define just one real value of α if the inequalities

$$\tau \leq t, \quad \xi'^2 + \eta'^2 + \zeta'^2 < c^2, \quad (\xi' = d\xi/d\tau),$$

are added. If, moreover, l, m, n, p are linear functions of β the second of equations (8) will define just one value of β and this will generally be a complex function of x, y, z, t. Let us choose the function $f(\alpha, \beta)$ in formulas (9) so that $f^{-1} = \beta - \beta_0$, where β_0 is an arbitrarily chosen complex quantity;* the components of M will then generally be infinite at points on the moving curve $\beta = \beta_0$.

Now when $\beta = \beta_0$ and α is kept fixed, θ has a definite complex value and so it follows that the line joining the points (x, y, z, t), (ξ, η, ζ, τ) is fixed in direction. Taking into account the first of equations (8) we see that the possible positions of (x, y, z, t) are those of a point which starts at (ξ, η, ζ, τ) and travels along a straight line with the velocity of light. By giving different real values to α we obtain a series of such points which form at any instant a singular curve of the æthereal field specified by (9). This singular curve is attached to the moving point (ξ, η, ζ, τ). If we give different complex values to β_0, a series of such curves is obtained, there being one for each value of β_0.

Let us now see if it is possible to choose the arbitrary functions at our disposal in such a way that the singular curve of an æthereal field of the type just considered is always a line of electric force in an electromagnetic field having the moving point (ξ, η, ζ, τ) as simple singularity.

Liénard has shown that an electromagnetic field with (ξ, η, ζ, τ) as a moving point charge may be derived from the potentials

$$(10) \qquad A_x = \xi'/\nu, \quad A_y = \eta'/\nu, \quad A_z = \zeta'/\nu, \quad \Phi = c/\nu,$$

where

$$\nu = \xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta) - c^2(t - \tau).$$

Writing

$$x - \xi = cl_0(t - \tau), \qquad y - \eta = cm_0(t - \tau),$$
$$x - \zeta = cn_0(t - \tau),$$
$$c - l_0\xi' - m_0\eta' - \eta_0\zeta' = \lambda, \qquad c^2 - \xi'^2 - \eta'^2 - \zeta'^2 = \mu,$$

* It may be a function of α but not of β.

we find on calculating the components of E by means of (2) that

$$E_x = \frac{1}{c\nu^2}\frac{\partial\tau}{\partial t}[\mu(\xi' - cl_0) + c(t-\tau)\{\lambda\xi'' + (\xi' - cl_0)(l_0\xi'' + m_0\eta'' + n_0\zeta'')\}].$$

Now let l_0, m_0, n_0 be regarded as functions of τ and let us suppose that particles are projected from the different positions of the singularity (ξ, η, ζ, τ) in directions specified by the direction cosines (l_0, m_0, n_0), and that they travel along straight lines with the velocity of light. If (x, y, z), $(x + dx, y + dy, z + dz)$ are the coordinates at time t of two particles that were projected at times τ, $\tau + d\tau$, we find that

$$\frac{dx}{d\tau} = \xi' - cl_0 + c(t-\tau)\frac{dl_0}{d\tau}.$$

Comparing this with the expression for E_x, we see that the line joining two consecutive particles coincides in direction with a portion of a line of electric intensity at each time t, if the derivatives of l_0, m_0, n_0 are given by equations of type

$$\mu\frac{dl_0}{d\tau} = \lambda\xi'' + (\xi' - cl_0)(l_0\xi'' + m_0\eta'' + n_0\zeta'').$$

If this condition is satisfied, the series of projected particles will form at each instant t a line of electric force in the electromagnetic field with the singularity (ξ, η, ζ, τ). By taking different initial directions for (l_0, m_0, n_0) we may obtain in this way all the lines of electric force.

Since the functions l, m, n, p are arbitrary as far as their dependence on α is concerned, we may evidently choose them so that the direction given by the equation $\beta = \beta_0$ coincides at each instant τ with the direction specified by the functions (l_0, m_0, n_0). This means that we can choose a system of æthereal fields in such a way that their singular curves coincide at each instant with the lines of electric force in an electromagnetic field having (ξ, η, ζ, τ) as simple singularity.

As regards the conditions of conjugacy, we have already proved that the electromagnetic field specified by the potentials (10) is conjugate to any field of type (9) in which α, β are the special functions considered in this paragraph.* We

*E, pp. 126, 133.

have also proved that a field of type (9) is self-conjugate* and is such that the flow of energy (Poynting's vector) is along the radius from (ξ, η, ζ, τ) to (x, y, z, t), hence it follows that the different æthereal fields given by (9) form a mutually conjugate system.

There are still many questions to be answered before the present theory can be accepted. If the æther is regarded as built up of elements a law describing the mutual action of two elements is needed. The nature of the macroscopic equations satisfied by the vectors (E, H) in a region occupied by an enormous aggregate of singular curves of æthereal fields has not been ascertained. The field obtained by superposing all the æthereal fields is probably different in character from an ordinary electromagnetic field; it may possibly have some connection with the phenomena of gravitation.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE,
October 27, 1914.

SHORTER NOTICES.

Handbuch des mathematischen Unterrichts. Von W. KILLING und H. HOVESTADT. Band II. Leipzig and Berlin, Teubner, 1913. x + 472 pp.

THE general character of this work was discussed by the reviewer in the BULLETIN, volume 17, No. 5. The vigor which characterized the first volume is unabated in the present one, which is devoted to trigonometry. One third of the volume is devoted to plane trigonometry, an equal amount to solid geometry or stereometry in which plane trigonometry is used freely in dealing with space problems, and the remainder to spherical trigonometry. There is sharp criticism of the ordinary text-books on trigonometry, in which the authors deplore the tendency to limit the field to goniometry and the solution of triangles with and without logarithms. Nor are they satisfied with a criticism of the content, for the methods of proof come under fire as well. As an illustration of the carefulness of their investigation, four proofs of the addition

* *E*, p. 12; *Phil. Mag.*, Jan., 1914.

theorem for the sine as given in different texts are stated, and thoroughly discussed as to their value, not merely in proving the immediate theorem, but in suggestiveness and applicability to further problems. The author of an ordinary American text-book would not feel flattered if he measured his product by the criteria of this volume. To show the value of trigonometry as a weapon in geometry, various well-known theorems such as the Feuerbach theorem are proved and an entire chapter of 50 pages is given to the application of trigonometry to construction problems. The chapter on teaching the applications of plane trigonometry is rich in suggestions, and could be read with profit by any teacher of the subject.

The section on stereometry covers a wider range than the usual elementary solid geometry. Among the applications is an interesting chapter on the geography of the sky and map making, including mercator and stereographic projection. Some chapters in fine print, such as the calculation of the surface of a curved solid by integration, and the bisection of a tetraedron by a hyperbolic paraboloid, are obviously of more interest to the teacher than to the immature student. The chapter on Euler's theorems giving the relations between the number of edges, faces, vertices, and face triangles of a convex polyedron is valuable, and shows forcibly how closely the authors have scrutinized supposed proofs. In talking about Riemann surfaces they again go beyond the depth of the ordinary student of solid geometry. The deficiency or redundancy of the numerous definitions of a regular polygon seems to be an object of keen attack. The chapter on regular spherical polygons in connection with regular polyedra is excellent and might well be incorporated into elementary texts.

The closing section on spherical trigonometry is no less critical than the earlier parts of the work, although in content it more nearly approximates a text-book on the subject. The proofs of Gauss's formulas and Napier's analogies are carefully considered, and the authors conclude their criticism of existing proofs by a new one of their own. They see no use in developing a mass of formulas which have historical interest only, so special theorems and formulas which play no rôle in the applications are reserved for the last chapter. The deduction of the theorems on the geometry of the sphere from the formulas of spherical trigonometry is a point which

teachers might well note. The applications considered are the ordinary ones,—measurement of the earth, astronomical problems, determining one's position at sea, and the like. The last chapter is a discussion of the work of Möbius on spherical triangles whose sides or angles may be greater than 180°, and of the still further extensions of Study. Any reader of this volume will be impressed with the possibilities of a course in trigonometry.

D. D. LEIB.

Théorie des Nombres. Par E. CAHEN. Tome premier: *Le premier Dégré.* Paris, A. Hermann et Fils, 1914. xii + 408 pp.

THE first eight chapters contain an exposition, mainly following the ideas of Helmholtz, of the definition and properties of integers, the four fundamental operations, divisibility, greatest common divisor and least common multiple, and the theory of fractions. Chapter 9 deals with diophantine equations in one and two unknown quantities, and the euclidean algorithm. Chapters 10 to 12 contain an exhaustive treatment of diophantine equations in any number of unknown quantities, and systems of such equations, preceded by an exposition of the corresponding algebraic theories (determinants and systems of linear equations). Chapters 13 to 16 give the theory of linear substitutions and linear and bilinear forms, the number theoretic case, where the coefficients are integers, being always preceded by an exposition of the corresponding algebraic case, where the coefficients are arbitrary. Chapter 17 is concerned with linear congruences, and chapters 18 to 20 contain the fundamentals of the algebra of matrices, followed by the corresponding number theoretic propositions for matrices with integers as elements. The three last chapters give the decomposition of an integer into prime factors, the properties of Euler's function $\varphi(n)$ and some other number theoretic functions, and some remarks on linear congruences with prime modulus.

Misprints are rather numerous, though seldom misleading, and it is difficult to see why the name of Frobenius should be persistently misspelled; one may also well take exception to the narrow and unusual definition of number theory proposed in the introduction. But these are minor criticisms, and the volume should prove of great value as a text-book, since the

exposition is remarkably clear and easy to follow, and illustrated by numerous and well-chosen numerical examples and exercises. Several topics, for instance the arithmetical reduction of bilinear forms, are presented here in a more lucid and accessible form than in any other work known to the reviewer.

T. H. GRONWALL.

Leçons sur la Théorie générale des Surfaces et les Applications géométriques du Calcul infinitésimal. Par GASTON DARBOUX. Première partie: *Généralités. Coordonnées curvilignes. Surfaces minima.* Deuxième édition, revue et augmentée. Paris, Gauthier-Villars, 1914. vii + 618 pp.

AFTER having been out of print for some time, the first volume of Darboux's classical treatise now appears in a second edition. Since the general features of this admirable work are undoubtedly familiar to the majority of the BULLETIN'S readers, it will be sufficient to mention, in this review, the new matter added in the second edition and taken mostly from Darboux's own papers in the *Bulletin des Sciences mathématiques* and the *Comptes rendus*.

In Book I, Chapters 5 and 6 deal with the kinematics of motions dependent on any number of parameters, and the integration of the corresponding total differential equations for the direction cosines. Chapter 7, dealing with the special case of two parameters, brings some new developments regarding the Plücker conoid. Chapter 10 gives a new solution, which is both elementary and elegant, of a problem first proposed and solved by Sophus Lie: to determine all surfaces which can be generated in more than one way by the translation of a rigid curve. It seems, however, to have escaped the notice of the distinguished author that a similar solution has been previously given by Scheffers ("Das Abelsche Theorem und das Liesche Theorem über Translationsflächen," *Acta Mathematica*, volume 28 (1904), pages 65–91).

In Book II, there has been added to Chapter 1 a study of a special conjugate system formed by plane curves, which leads to a class of surfaces applicable on quadric surfaces and discovered by Peterson. Chapter 4 contains an exposition of Gauss's method for the conformal representation of the terrestrial ellipsoid on a sphere, as well as a solution of a problem

proposed by Lagrange: to determine all geographical maps such that the meridians and parallels are represented by circles.

In Book III, Chapter 3 gives the determination of all minimal surfaces capable of being generated by translation in more than one way, and in Chapter 8 a special minimal surface is considered which was discovered by Riemann and is generated by the motion of a circle of variable radius. Finally, Chapter 9 gives an investigation of Ribaucour's isotropic congruences, with applications to Bertrand's curves.

T. H. GRONWALL.

NOTES.

THE opening (January) number of volume sixteen of the *Transactions of the American Mathematical Society* contains the following papers: "On the theory of curved surfaces, and canonical systems in projective differential geometry," by G. M. GREEN; "The multitude of triad systems on 31 letters," by H. S. WHITE; "The φ-subgroup of a group," by G. A. MILLER; "On a set of postulates which suffice to define a number-plane," by R. L. MOORE; "The equivalence of complex points, planes, lines with respect to real motions and certain other groups of real transformations," by W. C. GRAUSTEIN; "Invariants of the rational plane quintic curve and of any rational curve of odd order," by J. E. ROWE; "A set of postulates for general projective geometry," by M. G. GABA; "Certain quartic surfaces belonging to infinite discontinuous cremonian groups," by V. SNYDER and F. R. SHARPE; "The functions of a complex variable defined by an ordinary differential equation of the first order and the first degree," by J. SLEPIAN; "On the differential geometry of ruled surfaces in 4-space and cyclic surfaces in 3-space," by A. RANUM.

THE opening (January) number of volume 37 of the *American Journal of Mathematics* contains the following papers: "Generalizations of geodesic curvature and a theorem of Gauss concerning geodesic triangles," by G. A. BLISS; "On the medians of a closed convex polygon," by A. EMCH; "The

rational sextic curve, and the Cayley symmetroid," by J. R. CONNER; "Limited and illimited linear difference equations of the second order with periodic coefficients," by T. FORT; "Finite groups of plane birational transformations with eight fundamental points," by F. R. SHARPE; "Conics through inflections of self-projective quartics," by F. R. SHARPE; "Concerning an analogy between formal modular invariants and the class of algebraical invariants called Booleans," by O. E. GLENN; "Periodic orbits on a smooth surface," by D. BUCHANAN; "On Foucault's pendulum," by W. D. MACMILLAN.

The frontispiece of the volume is a portrait of JOHN NAPIER.

THE first number (January 15, 1915) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "Recent progress in the theories of modular and formal invariants and in modular geometry," by L. E. DICKSON; "The synthesis of triad systems Δ_t in t elements, in particular for $t = 31$," by H. S. WHITE; "The φ-subgroup of a group of finite order," by G. A. MILLER.

The second number (February 15) contains: "Transformations of surfaces Ω," by L. P. EISENHART; "Conjugate systems of space curves with equal Laplace-Darboux invariants," by E. J. WILCZYNSKI.

THE following papers on mathematics or mathematical physics have recently appeared in the *Proceedings of the American Academy of Arts and Sciences:* "The generalized Riemann problem for linear differential equations, and the allied problems for linear difference, and q-difference equations," by G. D. BIRKHOFF, volume 49, pages 519–568; "On the theory of the rectilinear oscillator," by E. B. WILSON, volume 50, pages 105–128; "Planck's radiation formula and the classical electrodynamics," by D. L. WEBSTER, volume 50, pages 128–145.

AT the meeting of the London mathematical society held December 10 the following papers were read: "Simultaneous equations, linear or functional," by E. H. NEVILLE; "Cyclotomic quinquisection," by W. BURNSIDE; "Oscillations near the isosceles triangle solution of the three body problem," by D. BUCHANAN; "Lamé's differential equation and ellipsoidal harmonics," by E. T. WHITTAKER.

THE Mathematical association held its annual meeting January 9 at the London day college. The following papers were read: "Mathematics in artillery science," by G. GREENHILL; "Teaching of modern analysis," by W. P. MILNE; "The circle of curvature," by A. LODGE; "Practical work in connection with mathematics," by R. C. FAWDRY.

A NOTICE has recently been sent out by Professor Fehr, secretary of the International commission on the teaching of mathematics, announcing that, on account of the war, it has been decided not only to abandon the idea of a meeting in August 1915, but also to postpone the preparation of such projected reports as relate to the European countries.

THE University of Chicago Press announces the following volumes in preparation for early publication in its Science Series: Finite collineation groups, by H. F. BLICHFELDT; Linear integral equations in general analysis, by E. H. MOORE.

AT Harvard University two Benjamin Peirce instructorships in mathematics have been established, the holders of which will devote a part of their time to teaching, including some advanced instruction, but will be given facilities and encouragement to prosecute scholarly research. Appointments are made, under open competition, for a term of one year, and may be renewed for not more than three years. Further information may be obtained from the chairman of the division of mathematics, PROFESSOR BÔCHER.

AT Cornell University provision has been made for giving two cycles of advanced courses in analysis and in geometry in the summer session, each cycle to run through four years. For 1915 these courses will begin with Advanced calculus and infinite series, by Professor W. A. HURWITZ, five hours, and Algebraic plane curves, by Professor VIRGIL SNYDER, five hours.

PROFESSOR DE LA VALLÉE POUSSIN, of the University of Louvain, will give a course of lectures in French at Harvard University on Lebesgue integrals, beginning February 16 and extending through the remainder of the academic year.

DR. M. BOTTASSO has been appointed docent in analytic geometry at the University of Pavia.

DR. T. H. HAVELOCK has been appointed professor of applied mathematics and mathematical physics at Armstrong College, Newcastle-on-Tyne.

PROFESSOR A. W. PHILLIPS, formerly dean of the Yale University graduate school, died January 20, at the age of 70 years. He became a tutor at Yale in 1877 and retired from his professorship in 1911. He was a member of the American Mathematical Society since 1896.

PROFESSOR A. VENTURI, of the University of Palermo, died December 29 at the age of 62 years.

Book catalogues: Deighton, Bell and Company, 13 Trinity Street, Cambridge, England, catalogue 27, some 300 mathematical pamphlets, theses, etc.—Gustav Fock, Schlossgasse 7, Leipzig, Handapparate of Amthor, Lüroth, Schubert, and others.—Julius Springer, Linkstrasse 23, Berlin, technical works, applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAYLISS (R. W.). A first school calculus. New York, Longmans, 1914. Cr. 8vo. 12+288 pp. $1.20

BLUEL (E.). Leçons de mathématiques spéciales. Volume 1: Algèbre, ligne droite et plan, trigonométrie, analyse, applications géométriques. Paris, 1914. 8vo. Fr. 15.00

——. Leçons de mathématiques spéciales. Volume 2: Géométrie analytique, courbes et surfaces. Paris, 1914. 8vo. Fr. 15.00

BÔCHER (M.). Charles Sturm et les mathématiques modernes. Paris, Alcan, 1914. 8vo. 19 pp.

——. Mathématiques et mathématiciens français. Paris, Champion, 1914. 8vo. 16 pp.

BOREL (E.). Leçons sur la théorie des fonctions. Eléments et principes de la théorie des ensembles; applications à la théorie des fonctions. 2e édition. Paris, Gauthier-Villars, 1914. 8vo. 12+260 pp. Fr. 7.50

CREMONA (L.). Elements of projective geometry. Translated by Charles Leudesdorf. 3d edition. Oxford, Clarendon Press, 1914. 8vo. 322 pp. 12s. 6d.

CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. 3te erweiterte Auflage. (2 Bände.) Band I: Wahrscheinlichkeitstheorie, Fehlerausgleichung, Kollektivmasslehre. Leipzig, Teubner, 1914. Gr. 8vo. 12+462 pp. Cloth. M. 14.00

HJELMSLEV (J.). Grundlag for flaudernes geometrie. Kjöbenhavn, Vid. Selsk. Skrift., 1914. 8vo. 52 pp.

HORT (W.). Die Differentialgleichungen des Ingenieurs. Berlin, Springer, 1914. 8vo. 9+540 pp. Cloth. M. 14.00

KOMMERELL (V.). Raumgeometrie (Stereometrie und darstellende Geometrie). 2te Auflage. Tübingen, H. Lauppsche Buchhandlung, 1914. 213 pp. M. 2.80

LEUDESDORF (C.). See CREMONA (L.).

LOEWY (A.). Lehrbuch der Algebra 2 Theile. Theil I: Grundlagen der Arithmetik. Leipzig, 1914. Gr. 8vo. M. 12.00

LOUP (W. M.). Bijdrage tot de theorie der cyclische oppervlakken en congruenties. Utrecht, 1914. 8vo. 88 pp.

MATHEMATISCHE ABHANDLUNGEN Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. August 1914 gewidmet von Freunden und Schülern. Berlin, Springer, 1914. 8vo. 8+451 pp. 53 figures. M. 24.00

MATHEWS (G. B.). Projective geometry. (Longmans Modern Mathematical Series). London, Longmans, 1914. 8vo. 14+349 pp. 5s.

PALM (R.). Zur Invariantentheorie eines Pfaffschen Ausdrucks. (Diss., Greifswald.) Leipzig, Teubner, 1914.

ROBB (A. A.). A theory of time and space. Cambridge, University Press, 1914. Royal 8vo. 6+373 pp. Cloth. 10s. 6d.

SCHATUNOVSKY (J.). Der grösste gemeinschaftliche Teiler von algebraischen Zahlen zweiter Ordnung mit negativer Diskriminante und die Zerlegung dieser Zahlen in Primfaktoren. Strassburg, 1912. 8vo. 67 pp.

SCHWARZ (H. A.). See MATHEMATISCHE ABHANDLUNGEN.

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). Elements of geometry. Parts 1–6. London, Macmillan, 1914. 8vo. 9+465+38 pp.+answers. 4s. 6d.

ČERVENKA (L.). See VOROVKA (K.).

CHILD (J. M.). See BARNARD (S.).

COOK (F. W.). See WILKINSON (P.).

CRAWLEY (E. S.). One thousand exercises in plane and spherical trigonometry. Philadelphia, E. S. Crawley, 1914. 8vo. 70 pp.

DANZER (O.). Die darstellende Geometrie am Realgymnasium. Wien, 1912. 8vo. 63 pp.

DOLINSKI (M.). Politische Arithmetik. Zinseszinsenrechnung und Versicherungsmathematik nebst zwei Logarithmentafeln. Wien, 1914. Gr. 8vo. 292+57 pp. M. 5.00

DUPUIS (J.). Tables de logarithmes à 7 décimales, contenant les logarithmes des nombres de 1 à 100,000 et des sinus et tangentes des angles calculés de seconde à seconde pour les 5 premiers degrès et de 10 secondes à 10 secondes pour tous les degrès du quart cercle. 46e tirage. Paris, 1914. 8vo. 11+581 pp. Fr. 9.00

ERIKSSON (E. V.). Om sectio anguli och om de af detta problem betingade algebraiska ekvationera. Göteborg, 1914. 8vo. 41 pp.+3 tables. M. 1.50

GILLESPIE (F. C.). Worked exercises in elementary geometry. Oxford, Blackwell, 1914. 8vo. 4+136 pp. 3s. 6d.

GUEZOUNI (C.). The trisection of lines and angles. London, International Scientific Association, 1914. 1s.

HOWELL (H. B.). A foundation study in the pedagogy of arithmetic. London, Macmillan, 1914. 5s. 6d.

JÄDERIN (E.). Tables abrégées pour le calcul des logarithmes à 12 décimales. Stockholm, 1914. 4to. 21 pp.

LONGMANS explicit arithmetics. London, Longmans, 1914. 8vo. Pupil's Book IV, 3d. Teacher's Book IV, 1s.

POSEJPOL (V.). See VOROVKA (K.).

RICHARDSON (S. F.). Solid geometry. Boston, Ginn, 1914. 8vo. 5+209 pp. $0.90

SAMSONOFF (J.). The trisection of an angle. New York, Publishers Printing Co., 1914. 8vo. 12 pp. $0.25

SANJANA (K. J.). First year course in mathematics. Geometry and trigonometry. Bombay, K. & J. Cooper, 1914. 240 pp.+15 tables. Ru. 1.12

SANKEY (E.). Examples in easy practical drawing for the second year preliminary technical course. Books 1–2 with answers. 66+66 pp. 8d.+8d.

SMITH (D. E.). See WENTWORTH (G.).

SUZZALO (H.). See WALSH (J. H.).

VOROVKA (K.), ČERVENKA (L.) und POSEJPOL (V.). Die Lehrbücher für Mathematik, darstellende Geometrie und Physik an den Mittelschulen mit böhmischer Unterrichtssprache. (Bericht über den math. Unterricht in Oesterreich.) Wien, Hölder, 1914. 89 pp.

WALKER (W. J.). Examples and test papers in algebra. Parts 2–3. London, Mills and Boon, 1914. 8vo. 8+163–338 pp. 1s. 6d.

WALSH (J. H.) and SUZZALO (H.). Three-book series of arithmetics. Boston, Heath, 1914. 12mo. Cloth. Fundamental processes. 6+246 pp. $0.35. Essentials. 6+290 pp. $0.40. Business and industrial practice. 9+366 pp. $0.45.

——. Two-book series of arithmetics. Boston, Heath, 1914. 12mo. Cloth. Fundamental processes. 6+246 pp. $0.35. Practical applications. 6+502 pp. $0.65.

WENTWORTH (G.) and SMITH (D. E.). Plane trigonometry and tables. Boston, Ginn, 1914. Royal 8vo. 6+188+6+104 pp. $1.10

WILKINSON (P.) and COOK (F. W.). Macmillan's reform arithmetic for rural schools. (Standards V to VII). London, Macmillan, 1914. 5d.

YATE-LEE (F. S.). An introductory geometry. Part 1 (for teachers only). 19 pp. Part 2. 32 pp. Ripon, Thirlway, 1914.

III. APPLIED MATHEMATICS.

BAUERMANN (E.). Untersuchungen über die Ausführbarkeit einer Methode, Kontaktpotentialdifferenzen durch ein trennendes Dielektrikum hindurch zu messen. Rostock, 1912. 8vo. 37 pp. M. 1.50

BOUASSE (H.). Cours de physique. 2e édition, complètement transformée et augmentée. Tome III: Magnétisme et électricité (3 parties). Partie 2: Applications. Paris, Delagrave, 1914. 8vo. 560 pp. Fr. 22.00

BONIN (H.). Experimentelle Bestimmung des Ungleichsförmigkeitsgrades. (Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, herausgegeben vom Verein Deutscher Ingenieure, Heft 165). Berlin, 1914. Lex. 8vo. 35 pp. M. 1.00.

BRUNNER (J. B.). Lehrbuch der Physik. Zürich, 1914. 8vo. 420 pp. +4 tables. Geb. M. 7.30

BÜRGIN (J.). Genauigkeitsuntersuchungen über die Bestimmung der Intensität der Schwerkraft durch relative Pendelmessungen auf 9 Stationen des Badischen Oberlandes und auf der Schweizerischen Referenzstation zu Basel. Heidelberg, 1912. 4to. 112 pp.+6 tables.

CARTE photographique du ciel (Observatoire de Toulouse). Coordonnées rectilignes. Tome IV: Zone +6° à +8°. Fascicule 2. Paris, 1913. 4to. Pp. 53–220. Fr. 23.00

CRABTREE (H.). An elementary treatment of the theory of spinning tops. London, Longmans, 1914. 8vo. 15+193 pp. 7s. 6d.

FLAMM (L.). Die neuen Anschauungen über Raum und Zeit. Das Relativitätsprinzip. Wien, Schrift. Ver. Verbr. nat. Kenntn., 1914. 8vo. 46 pp.

FLAMMARION (C.). Le système du monde et Copernic. Nouvelle édition. Paris, 1914. 8vo. Fr. 0.60

FOWLE (F. E.). Smithsonian physical tables. 6th revised edition. (*Smithsonian Miscellaneous Collections*, Vol. 63, No. 6). Washington, Smithsonian, 1914. 36+355 pp.

GIBSON (A. H.) and RITCHIE (E. G.). A study of the circular-arc bow-girder. London, Constable, 1914. 4to. 10s. 6d.

GRAETZ (L.). Die Elektrizität und ihre Anwendungen. 17te umgearbeitete und vermehrte Auflage. Stuttgart, 1914. Gr. 8vo. 16+748 pp. Cloth. M. 9.00

GRAFF (K.). Grundriss der geographischen Ortsbestimmung aus astronomischen Beobachtungen. Berlin, 1914. Gr. 8vo. 8+210 pp. M. 8.00

GREENWICH. Astronomical, magnetical and meteorological observations made at the Royal astronomical observatory, Greenwich, in the year 1912. London, 1914. Royal 4to. 21s.

HAAS (A. E.). Die Grundgleichungen der Mechanik, dargestellt auf Grund der geschichtlichen Entwicklung. Vorlesungen zur Einführung in die theoretische Physik. Leipzig, 1914. Gr. 8vo. M. 7.50

HALER (P. J.) and STUART (A. H.). First course in mathematics for technical students. London, Tutorial Press, 1914. 1s. 6d.

HAUSSNER (R.). Darstellende Geometrie. 1ter Teil, 3te Auflage. 207 pp. 2ter Teil, 2te Auflage. 168 pp. (Sammlung Göschen Nr. 142 u. 143.) Leipzig, Göschen, 1914. M. 0.90+0.90

HAVELOCK (T. H.). The propagation of disturbances in dispersive media. (Cambridge Tracts in Mathematics, No. 17.) Cambridge, University Press, 1914. 8vo. 3s. 6d.

HOBB (—.). Hobb's arithmetic of electrical measurements. Revised by A. Risdon Palmer. London, Murby, 1914. 1s.

HOUSDEN (C. E.). Rapid earthwork calculation. London, Longmans, 1914. Cr. 8vo. 31 pp. 2s.

IKLÉ (M.). See MCCREADY (K.).

INGHAM (A. E.). Machine construction and drawing. Book 2. London, Routledge, 1914. 12+180 pp. 3s.

KOLLERT (J.). Die Wechselstrommaschinen, Transformatoren und Motoren. Kurze Darstellung ihres Baues und ihrer Wirkungsweise. 2te erweiterte Auflage von "Riedel, Wechselstrommaschinen." Leipzig, 1914. Gr. 8vo. 7+159 pp. M. 5.00

LOW (D. A.). An introduction to machine drawing and design. New edition, rewritten, reillustrated and enlarged. London, Longmans, 1914. 8vo. 8+248 pp. 3s.

MCCREADY (K.). Sternbuch für Anfänger. Uebersetzt von Max Iklé. 8vo. Leipzig, Barth, 1913. 9+150 pp.+2 tables. Geb. M. 12.00

MARCHANT (T. P.). See MORGAN (J.).

MORGAN (J.), MARCHANT (T. P.) and WOOD (A. L.). The "Conway" manual, being a complete summary of all problems of navigation and nautical astronomy. London, Potter, 1914. 79 pp. 5s.

MÜLLER (O.). Graphisches Rechnen und die graphische Darstellung. Ausgabe B. Glauchau i. S., Streit, 1913. 64 pp. Geb. M. 2.00

PALMER (A R.). See HOBB.

PLANCK (M.). Dynamische und statische Gesetzmässigkeit. Leipzig, 1914. 8vo. 32 pp. M. 1.00

RIEDEL (—). See KOLLERT (J.).

RIEGLER (G.). Sonnen- und Mondfinsternisse und ihre Bedeutung für die Himmelsforschung, leichtfasslich dargestellt. Wien, 1914. 8vo. M. 2.00

RITCHIE (E. G.). See GIBSON (A. H.).

SPOONER (H. J.). Engineering workshop drawing. First steps in engineering drawing. London, Longmans, 1914. Oblong 4to. 8+120 pp. 2s.

STUART (A. H.). See HALER (A. J.).

TURNER (H. H.). Tables for facilitating the use of harmonic analysis. Oxford, Clarendon Press, 1913. 1s.

VILLAMIL (R. DE). Motion of liquids. London, Spon, 1914. 8vo. 14+210 pp. Cloth. 7s. 6d.

WALKER (T. L.). Crystallography: an outline of the geometrical properties of crystals. New York, McGraw-Hill, 1914. (London, Hill Publ. Co.) 8s. 4d.

WOOD (A. L.). See MORGAN (J.)